

KHOVANOV HOMOLOGY AND TIGHT CONTACT STRUCTURES

OLGA PLAMENEVSKAYA

ABSTRACT. Using the relation between Khovanov homology and the Heegaard Floer homology of branched double covers, we show how Khovanov homology can be used to establish tightness of branched double covers of certain transverse knots. We give examples of several infinite families of knots whose branched covers are tight for Khovanov-homological reasons, and show that some of these branched covers are not Stein fillable.

1. INTRODUCTION

The goal of this paper is to demonstrate how Khovanov homology can be used to establish tightness of certain contact structures. The contact manifolds we study are all branched double covers of transverse knots in standard contact S^3 ; our main tool is the relation between the Khovanov-homological invariant ψ [Pl1] and the Heegaard Floer contact invariant $c(\xi)$ [OS2] of the branched double cover of K . This relation was conjectured by the author in [Pl2] and proved by Lawrence Roberts [Ro]; the following theorem is a corollary of the results in [Ro].

Theorem 1.1. *Suppose that the spectral sequence [OS3] from the reduced Khovanov homology to the Heegaard Floer homology $\widehat{HF}(-\Sigma(K))$ of the branched double cover of K collapses at the E^2 stage, thus providing an isomorphism between the Khovanov and the Heegaard Floer homology. (In particular, this is true when K belongs to a quasi-alternating knot type [OS3].) Then $c(\xi) \neq 0$ whenever $\psi \neq 0$.*

We work with $\mathbb{Z}/2$ coefficients throughout; ψ denotes the version of the invariant from [Pl1] that lives in the reduced Khovanov homology with $\mathbb{Z}/2$ coefficients.

We will use Theorem 1.1 together with the following non-vanishing criterion for ψ :

Theorem 1.2. *If K is a transverse knot such that $sl(K) = s - 1$, then $\psi(K) \neq 0$. The converse is also true if K belongs to a quasi-alternating (or any $Kh_{\mathbb{Z}/2}$ -thin) smooth knot type. Here $s = s(K)$ stands for Rasmussen's invariant [Ra2].*

Corollary 1.3. *If K is a transverse representative of a quasi-alternating knot, the induced contact structure ξ_K on the branched double cover of K has $c(\xi) \neq 0$ if $sl(K) = \sigma - 1$, where σ is the signature of the knot (with the sign conventions such that $\sigma(\text{right trefoil}) = 2$). In particular, ξ_K is tight whenever $sl(K) = \sigma - 1$.*

Note that for any transverse knot $sl(K) \leq s - 1$ [Pl1, Sh], so the condition of Theorem 1.2 is equivalent to sharpness of this upper bound for the self-linking number.

The values of the maximal self-linking number for knots with 10 crossings or less are known [Ng]; combining those with the above results, we can in certain cases establish the existence of a tight contact structure on the branched double cover of a given smooth knot.

To demonstrate the efficiency of Khovanov homology in proofs of tightness, we need to find examples of transverse knots satisfying the hypothesis of Corollary 1.3 (or those of Theorems 1.1 and 1.2). Many of these are provided by quasipositive braids, but the corresponding contact structures are obviously Stein fillable and therefore tight. However, non-quasipositive braids with $sl(K) = s - 1$ do exist; in section 3, we give examples of several infinite families of such transverse knots. We also show that some of the corresponding contact structures are not Stein fillable (and thus the result is non-trivial). Some of our examples are transverse 3-braids; accordingly, their branched covers have open book decomposition of genus one whose tightness can be established by methods [Bal1, HKM1]. Unlike [Bal1, HKM1], our proofs work for transverse braids of arbitrary index, require no explicit calculations of the Heegaard Floer contact invariants, and are completely combinatorial (once Theorem 1.1 is in our hands). In fact, John Baldwin has recently written a computer program [Bal2] for determining whether ψ is non-zero for a given transverse braid. If the underlying link is quasi-alternating (or otherwise satisfies the collapsing condition of Theorem 1.1), we can establish tightness of the corresponding contact structure by a computer calculation.

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2. NON-VANISHING OF THE TRANSVERSE AND CONTACT INVARIANTS

Theorem 1.1 is essentially a corollary of Roberts's work [Ro]. However, it is not contained in Roberts's paper, and as the constructions of [Ro] require some care, we find it useful to review the results of [Ro] before giving a proof of Theorem 1.1. Our notation below is a bit different from that of [Ro]; we will also ignore a few minor details (e.g. for technical reasons one needs to add two extra strands to a given transverse braid, etc).

We first recall the construction of spectral sequence that relates the reduced Khovanov homology and the Heegaard Floer homology of the branched double cover [OS3]. The main result of [OS3] gives a link surgeries filtered chain complex $(C(K), D)$ whose associated spectral sequence converges to $\widehat{HF}(-\Sigma(K))$ and has the E^1 -term

$$E^1 = \bigoplus_{i \in \{0,1\}^n} \widehat{HF}(-\Sigma(K_i)),$$

where the sum is taken over all complete resolutions K_i of the knot (or link) K . (As usual, the components of $i = (i_1, \dots, i_n)$ stand for 0- and 1- resolutions of crossings of K .) The filtration of $C(K)$ is given by the flattened cube filtration $I = \sum_{j=1}^n i_j$; the differential does not decrease this I -filtration. The complex $(C(K), D)$ is constructed by counting holomorphic polygons in (a symmetric product of) a Heegaard diagram compatible with all link surgeries; as a vector space,

$$C(K) = \bigoplus_{i \in \{0,1\}^n} \widehat{CF}(-\Sigma(K_i)),$$

and the differential D^0 on the associated graded object is given by the sum of the usual Heegaard Floer boundary maps $d : \widehat{CF}(-\Sigma(K_i)) \rightarrow \widehat{CF}(-\Sigma(K_j))$. One shows that for each complete resolution of the knot K , $\widehat{HF}(-\Sigma(K_i))$ is precisely the component $CKh(K_i)$ of the reduced Khovanov complex. (In fact, each manifold $-\Sigma(K_i)$ is simply the connected

sum of a few copies of $S^1 \times S^2$.) The differential D_1 on $\bigoplus_{i \in \{0,1\}^n} \widehat{HF}(-\Sigma(K_i))$ given by sums of maps

$$\widehat{HF}(-\Sigma(K_i)) \rightarrow \widehat{HF}(-\Sigma(K'_i))$$

with $I(i') = I(i) + 1$; each such map on \widehat{HF} is induced by the surgery cobordism between $\widehat{HF}(-\Sigma(K_i))$ and $\widehat{HF}(-\Sigma(K'_i))$ that corresponds to the change of resolution relating K_i and K'_i . Under the correspondence $\widehat{HF}(-\Sigma(K_i)) = CKh(K_i)$, these maps $\widehat{HF}(-\Sigma(K_i)) \rightarrow \widehat{HF}(-\Sigma(K'_i))$ are shown to be equal to maps $CKh(K_i) \rightarrow CKh(K'_i)$ that form the differential on the reduced Khovanov complex. This implies that the E^1 -term of the complex $C(K)$ is the reduced Khovanov complex, and the E^2 -term is the reduced Khovanov homology.

The main idea of [Ro] is to endow the above complex $C(K)$ with an additional filtration induced by the binding of an open book decomposition of $\widehat{HF}(-\Sigma(K))$. More precisely, represent the knot K as a (transverse) braid in S^3 , and let B be the braid axis. We can always assume (stabilizing if necessary) that the braid index of K is odd. Then the manifold $\widehat{HF}(-\Sigma(K))$ has a natural open book decomposition with binding B and pages given by branched double covers of a disk; this open book is compatible with the contact structure on $\Sigma(K)$ induced by K . One can incorporate the knot B into constuctions of [OS3], i.e. consider link surgeries Heegaard diagrams compatible with B . Then the knot B induces the additional ‘‘Alexander’’ filtration on $C(K)$. In particular, B filters $E_I^1 = \bigoplus_{i \in \{0,1\}^n} \widehat{HF}(-\Sigma(K_i))$, and the cobordism maps $\widehat{HF}(-\Sigma(K_i)) \rightarrow \widehat{HF}(-\Sigma(K'_i))$ considered above respect the filtration. (It is important to note here that $\widehat{HF}(-\Sigma(K_i)) = \widehat{HFK}(-\Sigma(K_i), B)$, i.e. the knot homology is trivial and only gives a filtration of $\widehat{HF}(-\Sigma(K_i))$ in this case.) We keep the notation $C(K)$ for the bi-filtered complex, and use the subscripts A and I to distinguish between spectral sequences associated to different filtrations.

Roberts shows that as a bi-filtered complex, the term E_I^1 is isomorphic (modulo some grading adjustments) to the bi-filtered ‘‘skein Khovanov complex’’ [APS], which is the usual reduced Khovanov complex $\bigoplus_{i \in \{0,1\}^n} CKh(K_i)$ endowed with an extra filtration. The extra filtration is induced by B and comes from dividing the components of each resolution K_i of K into those circles that link with B and those that do not. It turns out that for each i , this filtration coincides with the Alexander filtration on knot Floer homology $\widehat{HF}(-\Sigma(K_i)) = \widehat{HFK}(-\Sigma(K_i), B)$, thus the skein Khovanov complex is indeed the E_I^1 -term for $C(K)$.

As before, we can consider the spectral sequence induced on $C(K)$ by the I -filtration; we still have $E_I^2 = Kh(K)$, and $E_I^\infty = \widehat{HF}(-\Sigma(K))$, but now each page of the spectral sequence inherits a filtration from the A -filtration on the original complex. (For a filtered chain complex, a filtration on its homology group is defined, as usual, by taking the minimum of filtration levels of cycles representing a given homology class.) We point out a caveat here: the A -filtration does not behave well with respect to the spectral sequence; in particular, when $E_I^n = E_I^\infty = \widehat{HF}(-\Sigma(K))$, the A -filtration on E_I^n computed from the spectral sequence may differ from the A -filtration on $\widehat{HF}(-\Sigma(K))$ computed by taking the homology of the entire complex (see Remark 2.1 below).

On the other hand, we can ignore the I -filtration and consider the spectral sequence from $C(K)$ to $\widehat{HF}(-\Sigma(K))$ induced by the A -filtration. Then the E_A^1 -term is given by $\widehat{HFK}(-\Sigma(K), B)$, since the link surgeries construction of [OS3] work just as well when we pass to the associated graded object for the knot filtration of B [Ro, Proposition 7.1]. Moreover, the subsequent pages of this spectral sequence are quasi-isomorphic to the pages of the knot Floer homology spectral sequence induced by the knot filtration of B on the Heegaard Floer complex $\widehat{CF}(-\Sigma(K))$ [Ro, Lemma 7].

Now, recall that the contact invariant $c(\xi)$ is the image in $\widehat{HF}(-\Sigma(K))$ of the unique lowest A -filtration element $c \in \widehat{HFK}(-\Sigma(K), B)$ (which lies in the $A = -g(B)$ filtration level). Accordingly, the $A = -g(B)$ filtration level in $\widehat{HF}(-\Sigma(K))$ is empty or one-dimensional depending on whether $c(\xi)$ vanishes or not. In the A -filtered skein $CKh(K)$, we can also pinpoint the unique lowest A -degree generator $\psi \in \bigoplus_{i \in \{0,1\}^n} CKh(K_i)$. Indeed, the construction of the skein filtration on $CKh(K)$ implies that the lowest A -degree can be only reached when we take the oriented resolution of the braid K (so that all resulting circles link with B), and pick the lowest quantum degree element $\psi = \mathbf{v}_- \otimes \cdots \otimes \mathbf{v}_-$ in the corresponding component $CKh(K_i)$. Observe that this is precisely the cycle in the reduced Khovanov complex that gives the transverse invariant $\psi \in Kh(K)$ of [P11]. (Strictly speaking, in the reduced case we take $\psi = \mathbf{v}_- \otimes \cdots \otimes \mathbf{v}_- \oplus \mathbf{v}_+$, with the \mathbf{v}_+ on the marked circle, but we may keep the notation without \mathbf{v}_+ by identifying the reduced complex with the subcomplex $CKh_-(K)$, see [P11] for details.) It then follows that in the (A -filtered) $E_I^2 = Kh(K)$ -term of the spectral sequence induced on $C(K)$ by the I -filtration, the $A = -g(B)$ filtration level is empty or one-dimensional depending on whether ψ vanishes or not.

Remark 2.1. Even though both $\psi(K) \in E_I^2$ and $c(\xi_K) \in E_I^\infty$ lie in the same lowest A -filtration level and are the images under the spectral sequence of the same canonical cycle in the skein Khovanov complex E_I^1 , one should use caution when talking about the “correspondence” between ψ and $c(\xi)$. Indeed, as was pointed out by John Baldwin, it is possible that $\psi(K) = 0$ while $c(\xi_K) \neq 0$. This happens when ψ is the boundary of a cycle x in E_I^1 , and Dx has other terms of higher filtration level in the entire chain complex, so that $c(\xi)$ is not a boundary. A toy example of this phenomenon is given by a complex (C, d) generated by three elements $x_{0,0}, y_{-2,1}, z_{-1,2}$, where the indices indicate the (A, I) bi-filtration, with $dx = y + z$. The lowest A -filtration element y plays the role of $c(\xi)$. In the I -induced spectral sequence, the differential d_0 is trivial, E_I^1 is generated by classes $[x], [y], [z]$, and $d^1[x] = [y]$. Thus $[y]$ is a boundary in E_I^1 , while y is not a boundary in the entire complex (C, d) . We also observe that even though the spectral sequence collapses at the E^2 -term, E_I^2 and $H_*(C, d)$ are different as A -filtered vector spaces: the filtration level of the generator $[z]$ of E_I^2 is $A = -1$, while the filtration level of the generator of $H_*(C, d)$ is $A = -2$.

(Baldwin found an explicit family of transverse braids with $\psi(K) = 0$ and $c(\xi_K) \neq 0$; we return to his examples in the next section.) However, we will show below that when the spectral sequence for the I -filtration collapses at the E^2 -term, non-vanishing of ψ implies non-vanishing of $c(\xi)$.

Proof of Theorem 1.1. The proof becomes easy if the chain complex $(C(K), D)$ satisfies the additional condition

- (*) the lowest A -filtration level of the complex $C(K)$ is $A = -g$;
 there is a unique generator c with $A(c) = -g$.

If (*) holds, the element c is necessarily a D -cycle concentrated in a single I -grading level, and thus gives rise to a cycle in every term of the spectral sequence. Moreover, because $A(\psi) = -g$ for the transverse element $\psi \in E_I^1$, c is a representative of the class ψ , and $I(c) = 0$.

To ensure that (*) holds, we may need to pass to a new complex $(C'(K), D')$, using a cancellation lemma (Lemma 2.2 below). In fact, applying this lemma to $(C(K), D)$ we obtain the complex $C'(K) = \bigoplus_{i \in \{0,1\}^n} \widehat{HFK}(-\Sigma(K_i), B)$. (Note that because $\widehat{HF}(-\Sigma(K_i)) = \widehat{HFK}(-\Sigma(K_i), B)$, the differential $(D')_I^0$ is trivial, so that the complex $C'(K)$ is the same as the E_I^1 -term for $C(K)$.) It is now clear that $C'(K)$ has a unique lowest A -filtration element c ; this element has $A = -g$ and lies in the component $\widehat{HFK}(-\Sigma(K_i), B)$ corresponding to the oriented resolution of the braid K .

Condition (4) of Lemma 2.2 implies that $\psi \neq 0$ if and only if E^2 -term of the I -induced spectral sequence on $(C'(K), D')$ is non-empty in the filtration level $A = -g$, and $c(\xi) \neq 0$ if and only if the E_I^∞ -term of $(C'(K), D')$ is non-empty in the same filtration level, because the same is true for $(C(K), D)$. We will now assume that $(C(K), D)$ satisfies (*).

Now, suppose that $c(\xi) = 0$. Then the $A = -g$ filtration level of $H_*(C(K), D)$ is empty, and thus the cycle c is a boundary, $c = Dx$ for some $x \in C(K)$. Because D is non-decreasing on the I -filtration, $I(x) \geq 0$. We claim that $I(x) > 0$: otherwise for the class $[x]$ in the associated graded object E_I^0 we have $D^0[x] = [Dx \text{ modulo terms of filtration level } I < 0] = [c]$, which contradicts the fact that the class of c (in E_I^1) is the non-zero cycle ψ . Thus $I(x) > 0$, and $D^0[x] = 0$, which means that $[x]$ is a cycle that gives rise to an element of E_I^1 . Next, consider $D^1[x]$. If $I(x) = 1$, then $D^1[x] = [Dx \text{ modulo terms of filtration level } I < 0] = [c] = \psi$, which contradicts the hypothesis that the transverse element $\psi \in E_I^1$ is not a boundary in the Khovanov skein complex (E_I^1, D^1) . Thus we conclude that $I(x) \geq 2$, and $D^1[x] = 0$, so that the class of $[x]$ is a cycle in E_I^1 giving rise to an element of E_I^2 . But since $Dx = c$, and $I(c) \leq I(x) - 2$, it follows that the spectral sequence does not collapse at the E_I^2 -term, a contradiction. \square

To state the cancellation lemma used above, we consider a finitely generated bi-filtered complex (C, d) with an ascending filtration I and a descending filtration A ,

$$\begin{aligned} I : \quad & \dots \subset C_{i+1} \subset C_i \subset \dots \\ A : \quad & \dots \subset C_{a-1} \subset C_a \subset \dots, \end{aligned}$$

and let $C_{i,a} = (C_i \cap C_a) / (C_{i+1} \cup C_{a-1})$ denote the bi-filtered quotients. Let $d_{00} : C_{i,a} \rightarrow C_{i,a}$ stand for the differential induced by d .

Lemma 2.2. ([Ra1, Lemma 4.5], [Ro, Lemma 8]) *Let C be a finitely generated bi-filtered complex (over $\mathbb{Z}/2$). There exists a unique (up to isomorphism) bi-filtered complex C' such that*

- (1) C and C' are bi-filtered chain homotopy equivalent

- (2) $(C')_{i,a} = H_*(C_{i,a})$
- (3) The differential d'_{00} on the quotients of C' is trivial.
- (4) For each filtration, the spectral sequences for (C, d) and (C', d') have the same terms, i.e.

$$E_I^r = (E'_I)^r \text{ and } E_A^r = (E'_A)^r \text{ for all } r \geq 0.$$

□

Proof of Theorem 1.2. Suppose that K is a transverse knot with $sl = s - 1$. We are interested in the reduced Khovanov homology with $\mathbb{Z}/2$ coefficients, but it is convenient to consider the case of non-reduced homology with rational coefficients first. (Our notation will often be the same for different flavors of Khovanov homology; it will be clear for the context which case we are considering.) Recall that Lee [Lee] introduces a differential $d' = d + \Phi$ on the Khovanov complex $CKh(K)$, where d is the usual Khovanov's differential [Kh], and Φ is a map that raises the quantum grading. We recall that d' corresponds to the multiplication and co-multiplication maps given by

$$\begin{aligned} m(\mathbf{v}_+ \otimes \mathbf{v}_+) &= m(\mathbf{v}_- \otimes \mathbf{v}_-) = \mathbf{v}_+ & \Delta(\mathbf{v}_+) &= \mathbf{v}_+ \otimes \mathbf{v}_- + \mathbf{v}_- \otimes \mathbf{v}_+ \\ m(\mathbf{v}_+ \otimes \mathbf{v}_-) &= m(\mathbf{v}_- \otimes \mathbf{v}_+) = \mathbf{v}_- & \Delta(\mathbf{v}_-) &= \mathbf{v}_- \otimes \mathbf{v}_- + \mathbf{v}_+ \otimes \mathbf{v}_+ \end{aligned}$$

(we follow the notation from [Ra2] where \mathbf{v}_- and \mathbf{v}_+ stand for the elementary generators of quantum degree -1 resp. $+1$.) This gives rise to a filtration in the Khovanov complex and yields a spectral sequence whose E^2 -term is the usual $Kh(K)$, and the E^∞ term is $Kh' = H_*(CKh, d')$. For a knot K , Lee's homology $Kh'(K) = \mathbb{Q} \oplus \mathbb{Q}$ is generated by two canonical cycles. Let \mathfrak{s}_0 be a canonical generator corresponding to a choice of orientation of K ; then ψ is a q -homogeneous part of \mathfrak{s}_0 with the lowest q -grading. Indeed, the invariant ψ is defined by representing the transverse knot by a braid, taking the oriented resolution, and taking the cycle

$$\psi = \mathbf{v}_- \otimes \mathbf{v}_- \otimes \mathbf{v}_- \dots$$

to be the lowest quantum degree term of the corresponding component of $CKh(K)$. The oriented resolution of a braid K consists of nested circles, and

$$\mathfrak{s}_0 = (\mathbf{v}_- + \mathbf{v}_+) \otimes (\mathbf{v}_- - \mathbf{v}_+) \otimes (\mathbf{v}_- + \mathbf{v}_+) \dots$$

is an element of CKh obtained by labeling these circles by $\mathbf{v}_- + \mathbf{v}_+$ and $\mathbf{v}_- - \mathbf{v}_+$, in alternating order. (The label on the outermost circle is determined by the orientation of the knot.) Rasmussen [Ra2] defines a function s on Kh' whose value $s(x)$ on $x \in Kh'$ is the largest n such that x can be represented by a cycle all of whose terms have quantum grading at least n . The invariant s is then defined so that $s - 1 = s_{\min} = s([\mathfrak{s}_0])$. Recall that $q(\psi) = sl(K)$ [Pl1], so our assumption means that $q(\psi) = s - 1$. Suppose that $\psi(K)$ vanishes in $Kh(K)$, so $\psi = dy$ for some $y \in CKh(K)$. Since d preserves quantum gradings, we must have $q(y) = s - 1$. Consider the element $\mathfrak{s}_0 - d'y$, where d' is Lee's differential on CKh . This is a cycle in Kh' which is homologous (in Kh') to \mathfrak{s}_0 and consists of terms with quantum grading $q > s - 1$, which contradicts the equality $s([\mathfrak{s}_0]) = s - 1$.

In the case of $\mathbb{Z}/2$ coefficients, Lee's theory as above does not produce a spectral sequence (indeed, the resulting homology is isomorphic to $Kh_{\mathbb{Z}/2}(K)$). However, a modification of Lee's construction [Tu] works in this case: one considers a filtered theory with

multiplication and comultiplication maps

$$\begin{aligned} m(\mathbf{v}_+ \otimes \mathbf{v}_+) &= \mathbf{v}_+ & \Delta(\mathbf{v}_+) &= \mathbf{v}_+ \otimes \mathbf{v}_- + \mathbf{v}_- \otimes \mathbf{v}_+ + \mathbf{v}_+ \otimes \mathbf{v}_+ \\ m(\mathbf{v}_- \otimes \mathbf{v}_-) &= \mathbf{v}_- & \Delta(\mathbf{v}_-) &= \mathbf{v}_- \otimes \mathbf{v}_- \\ m(\mathbf{v}_+ \otimes \mathbf{v}_-) &= m(\mathbf{v}_- \otimes \mathbf{v}_+) = \mathbf{v}_-. \end{aligned}$$

As explained in [Tu], Lee's arguments go through to yield a spectral sequence whose E^2 -term is the Khovanov homology with $\mathbb{Z}/2$ -coefficients, and the E^∞ -term is $\mathbb{Z}/2 \oplus \mathbb{Z}/2$ when K is a knot. When the knot K is given by a braid, the canonical generators for this theory are given by two elements $(\mathbf{v}_- + \mathbf{v}_+) \otimes \mathbf{v}_- \otimes (\mathbf{v}_- + \mathbf{v}_+) \dots$ and $\mathbf{v}_- \otimes (\mathbf{v}_- + \mathbf{v}_+) \otimes \mathbf{v}_- \dots$ obtained by labeling the alternate components of the canonical resolution of the braid by $(\mathbf{v}_- + \mathbf{v}_+)$ and \mathbf{v}_- . The transverse invariant ψ is again the lowest quantum degree part of the canonical generators. Moreover, a variant of the s -invariant can be defined in the same way, and by [MTV] it takes the same values as the original Rasmussen's s , so our argument from the preceding paragraph still applies.

It remains to deal with the reduced case. To obtain the reduced Khovanov complex, one places a marked point on the knot, forms the subcomplex $CKh_-(K)$ by labeling the the marked circle by \mathbf{v}_- in every resolution, and considers the quotient complex $CKh(K)/CKh_-(K)$. For $\mathbb{Z}/2$ coefficients, the spectral sequence of [Tu] works just as well in the reduced case. There is only one canonical generator \mathfrak{s}_o that survives; for a braid, it is given by the cycle $(\mathbf{v}_- + \mathbf{v}_+) \otimes \mathbf{v}_- \otimes (\mathbf{v}_- + \mathbf{v}_+) \otimes \dots = \mathbf{v}_+ \otimes \mathbf{v}_- \otimes (\mathbf{v}_- + \mathbf{v}_+) \otimes \dots$ with the label of $(\mathbf{v}_- + \mathbf{v}_+)$ on the marked circle of the oriented resolution of the braid. The lowest quantum degree part is $\mathbf{v}_+ \otimes \mathbf{v}_- \otimes \dots \mathbf{v}_-$, which is precisely the reduced version of the transverse invariant. The quantum grading shifts by 1 in the reduced case: we have $s^{red}(\mathfrak{s}_o) = s_{min}^{red} = s(K)$, and the transverse invariant lives in the component $Kh^{0,sl+1}$ of the reduced homology. This does not affect the validity of our argument.

To show the converse, note that for a $Kh_{\mathbb{Z}/2}$ -thin link, reduced $Kh^{0,*}$ can only be non-trivial for one value of the quantum grading, namely $q = s$. For the reduced version of the transverse invariant, $q(\psi) = sl + 1$, and the result follows.

(It is perhaps worth pointing out that the non-reduced Khovanov homology over $\mathbb{Z}/2$ is the direct sum of two copies of the reduced homology, $Kh_{\mathbb{Z}/2}^{n,q} = Kh_{\mathbb{Z}/2,red}^{n,q-1} \oplus Kh_{\mathbb{Z}/2,red}^{n,q+1}$ [ORS]. Thus, if we define thin knots as those whose non-reduced homology is supported on two diagonals, or those whose reduced homology is supported on one diagonal, the set of thin knots will be the same in both reduced and non-reduced cases.) \square

Proof of Corollary 1.3. We only need to recall two facts: by [MO], quasi-alternating knots are $Kh_{\mathbb{Z}/2}$ -thin and have $s = \sigma$, and by [OS2], non-vanishing of the contact invariant $c(\xi)$ is sufficient for tightness of a contact structure ξ . \square

3. EXAMPLES

In this section we give examples of contact structures whose tightness can be established by using Corollary 1.3. Since the result would be trivial for quasipositive braids, we are looking for non-quasipositive knots such that the s -bound for their self-linking number is nevertheless sharp. Among knots with 10 crossings or less, there are exactly three such knots, namely the mirrors of 10_{125} , 10_{130} and 10_{141} in the Rolfsen table. (As was indicated to the author by Lenny Ng, this can be seen by contrasting the list of quasipositive knots

from [Baa] and the values of the maximal self-linking numbers [Ng].) We use each of these knots to obtain an infinite family of tight contact structures, and show that contact structures in two of these families are not Stein fillable.

Example 3.1. For $r \geq 5$, consider the pretzel link $P(-r, 3, -2)$ (for $n = 5$, this is the mirror of the knot 10_{125} in the Rolfsen table), and let K_r be its transverse representative given by the closed braid

$$(\sigma_1)^{-r} \sigma_2 \sigma_1^3 \sigma_2.$$

We can use the algorithm from [HKP] to obtain the contact surgery description for the induced contact structure ξ on the branched double cover $\Sigma(K_r)$. For $r = 5$, we get the surgery diagram shown on the left of Figure 1; when $r > 5$, we have the diagram with r $(+1)$ -surgeries instead of five. (Strictly speaking, [HKP] gives a slightly different surgery diagram shown on the right of Figure 1. The two unoriented surgery links can be easily shown to be Legendrian isotopic, and we prefer the more symmetric diagram. In other examples below, we will also pick surgery links slightly different from but Legendrian isotopic to those given by [HKP].)

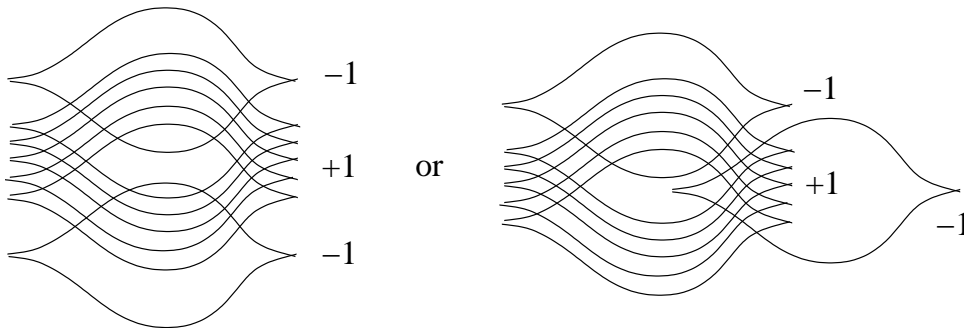


FIGURE 1. The surgery diagram for the branched double cover of the transverse knot $K = K_5$.

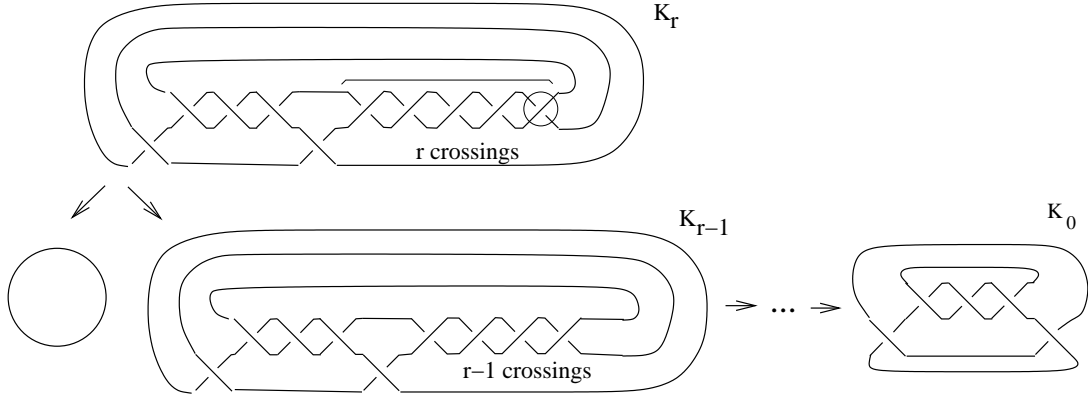
The underlying smooth manifold $\Sigma(K_r)$ is the Seifert fibered space $M(-1; 2/3, 1/2, 1/n)$. (See Figure 3 for a sequence of Kirby calculus moves demonstrating this for $n = 5$.)

Each link K_r is quasi-alternating. Indeed, $|\det(K_r)| = |H_1(M(-1; 2/3, 1/2, 1/n))| = r + 6$. On the other hand, resolving the crossing circled in Figure 2 in two possible ways, we obtain the link K_{r-1} and the unknot. Repeating the procedure r times, we get the link K_0 , which is the trefoil linked once with the unknot. Thus K_0 is an alternating link with $|\det(K_0)| = 6$, and, since $|\det(K_r)| = |\det(K_{r-1})| + |\det(\text{unknot})|$, we see by induction that K_r is quasi-alternating.

We next check the hypothesis of Theorem 1.2 when r is odd (i.e. K_r is a knot). We compute $sl(K) = 2 - r$. The knot K_r is Kh -thin, so s equals to the signature $\sigma(K_r) = 3 - r$ (we compute the signature via the Goeritz matrix of the knot [GL]).

When r is even, K_r is a two-component link, so Theorem 1.2 does not apply. However, we can argue that $\psi(K_r) \neq 0$ by [P11, Theorem 4], since $\psi(K_{r-1}) \neq 0$, and the transverse braid K_{r-1} is obtained from K_{r-1} by resolving a negative crossing.

Theorem 1.1 now implies that the branched double cover of each K_r is a tight contact manifold. We now show that none of them are Stein fillable. Since $\Sigma(K_5)$ can be obtained


 FIGURE 2. The links K_r are quasi-alternating.

from any of $\Sigma(K_r)$ by a sequence of Legendrian surgeries, it suffices to consider the contact structure $\xi_K = \xi$ that corresponds to $K = K_5$ and is shown on Figure 1.

We have already mentioned that the branched double cover of K is the Seifert fibered space $Y = M(-1; 2/3, 1/2, 1/5)$. Tight contact structures on this space were classified in [GLS]; Y carries three tight contact structures ξ_1 , ξ_2 and Ξ given by surgery diagrams on Figure 4. To identify our contact structure ξ among these three, we compute their d_3 invariants.

Recall [DGS] that the three-dimensional invariant d_3 of a contact structure given by a contact surgery diagram can be computed as

$$d_3(\xi) = \frac{c_1(\mathfrak{s})^2 - 2\chi(X) - 3\text{sign}(X) + 2}{4} + m,$$

where X is a 4-manifold bounded by Y and obtained by adding 2-handles to B^4 as dictated by the surgery diagram, \mathfrak{s} is the corresponding Spin^c structure on X , and m is the number of $(+1)$ -surgeries in the diagram. The Spin^c structure \mathfrak{s} arises from an almost-complex structure defined in the complement of a finite set in X , and the class $c_1(\mathfrak{s})$ evaluates on each homology generator of X corresponding to the handle attachment along an (oriented) Legendrian knot as the rotation number of the knot.

For the contact structure ξ on $Y = M(-1; 2/3, 1/2, 1/5)$ defined by the surgery diagram from Figure 1, we compute $c_1(\mathfrak{s}) = 0$ and $d_3(\xi) = -\frac{1}{2}$.

Let α be a Seifert surface a component of the Legendrian surgery link capped off by the core of a handle attached along this component; $c_1(\mathfrak{s})$ evaluates on α as the rotation number of the corresponding Legendrian knot. The classes of such surfaces α generate $H_2(X)$; labelling the components of Legendrian surgery links on Figure 4 as shown, we compute the Poincaré duals:

$$\begin{aligned} PDc_1(\xi_1) &= \frac{1}{11} (-29\alpha_1 - 29\alpha_2 + 20\alpha_3 + 12\alpha_4 - 3\alpha_5 + 6\alpha_6), \\ PDc_1(\xi_2) &= \frac{1}{11} (-17\alpha_1 - 17\alpha_2 + 14\alpha_3 + 4\alpha_4 - \alpha_5 - 2\alpha_6), \\ PDc_1(\Xi) &= \alpha_1 + \alpha_2 - \alpha_5, \end{aligned}$$

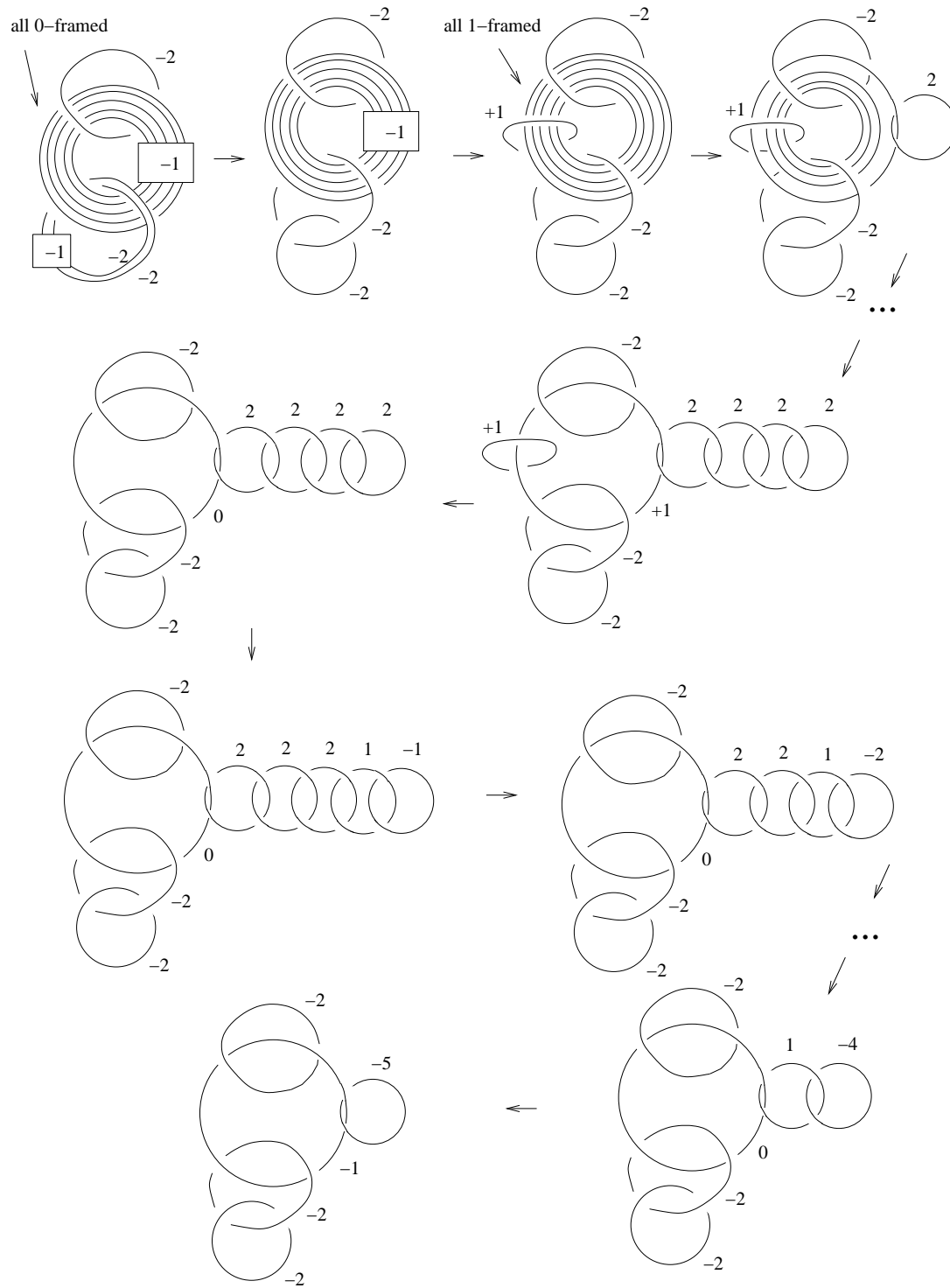


FIGURE 3. Kirby moves.

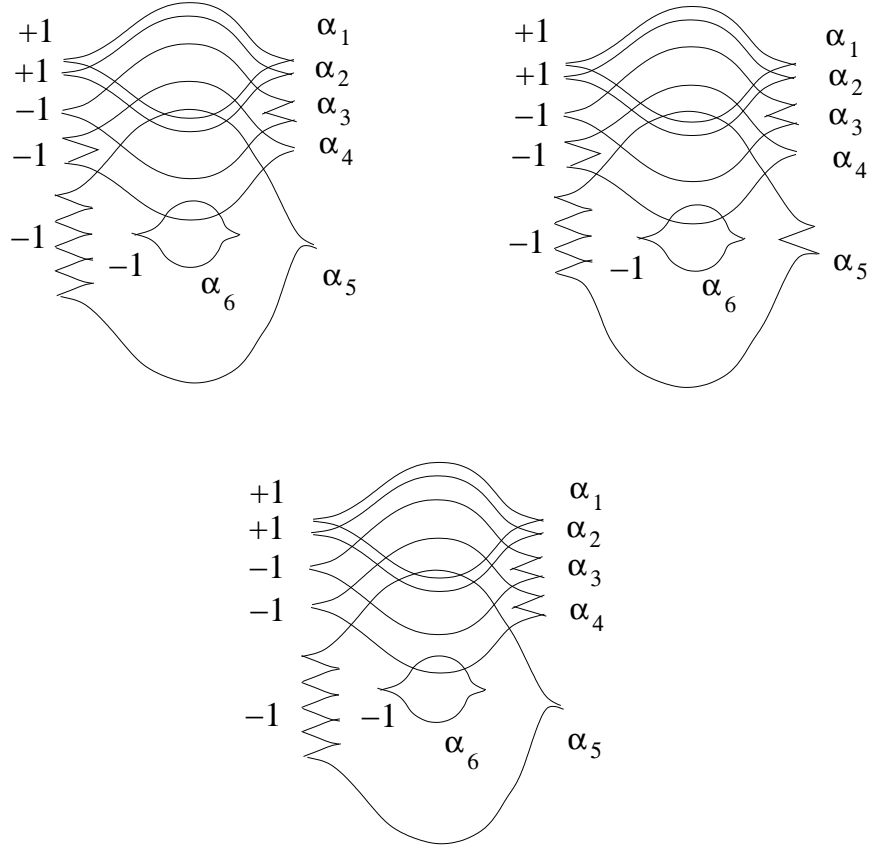


FIGURE 4. The manifold $Y = M(-1; 2/3, 1/2, 1/5)$ carries three tight contact structures: ξ_1 (top left), ξ_2 (top right), and Ξ (bottom).

and thus

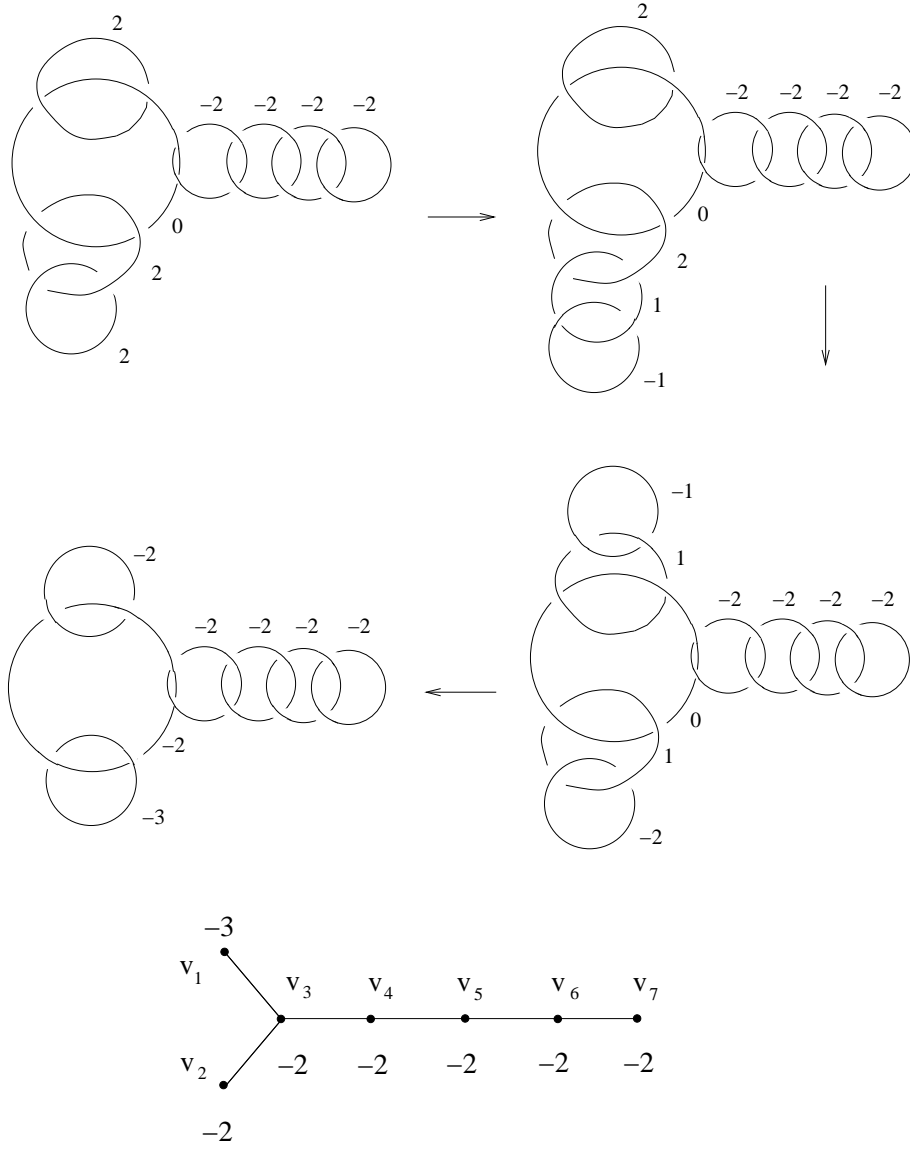
$$d_3(\xi_1) = \frac{1}{22}, d_3(\xi_2) = \frac{5}{22}, d_3(\Xi) = -\frac{1}{2}.$$

Because for the contact structure ξ from Figure 1 we have $d_3(\xi) = -\frac{1}{2}$, it follows that ξ is in fact the contact structure Ξ .

We show that ξ is not Stein fillable, combining the ideas from [GLS] and [Li]. More precisely, we will show that Y carries no Stein fillable contact structures with $d_3 = -\frac{1}{2}$.

We first observe that Y is an L -space, for example because it is a branched double cover of a quasi-alternating knot [OS3]. It follows [OS1] that $b_2^+(X) = 0$ for any symplectic filling X of a contact structure on Y . By the argument in [GLS], this implies that $b_1(X) = 0$. Now, observe that the space $-Y$ can be represented as the boundary of the plumbing shown on Figure 5.

Denote by W the 4-manifold with boundary $-Y$ given by this plumbing. If X is a symplectic filling for ξ , then $X \cup W$ is an oriented negative-definite closed 4-manifold. By Donaldson's theorem, the intersection form on $X \cup W$ is standard diagonal $\langle -1 \rangle^n$. To get restrictions on the intersection form of X , we consider the embeddings of the lattice given by Figure 5 into the standard negative-definite lattice, following [Li]. Let $e_i, i = 1, 2, \dots, n$,

FIGURE 5. Kirby diagrams for $-Y$ and the plumbing graph for W .

be the basis of $\langle -1 \rangle^n$ such that $e_i \cdot e_j = -\delta_{ij}$. Let v_i be the basis of $H_2(W)$ corresponding to the vertices of the plumbing graph of Figure 5. Up to permutations and sign reversals of e_i (which are automorphisms of the lattice $\langle -1 \rangle^n$), we have

$$\begin{aligned} v_3 &\mapsto e_1 + e_2, & v_2 &\mapsto -e_1 + e_3, & v_1 &\mapsto -e_1 - e_3 + e_5, \\ v_4 &\mapsto -e_2 + e_4, & v_5 &\mapsto -e_4 + e_6, & v_6 &\mapsto -e_6 + e_7, & v_7 &\mapsto -e_7 + e_8 \end{aligned}$$

(Another possibility would be for the first four vectors to embed as

$$v_3 \mapsto e_1 + e_2, \quad v_2 \mapsto -e_1 + e_3, \quad v_1 \mapsto -e_2 + e_4 + e_5, \quad v_4 \mapsto -e_1 - e_3,$$

but this leads to a contradiction when we try to embed v_5 .)

The orthogonal complement L of the image of the lattice generated by images of v_i 's in $\langle -1 \rangle^n$ is then spanned by the vectors

$$-e_1 + e_2 - e_3 + e_4 - 2e_5 + e_6 + e_7 + e_8, \quad e_9, \quad \dots \quad e_n,$$

and the intersection form on L is the diagonal form $\langle -11 \rangle \oplus \langle -1 \rangle^{n-8}$. Because $H_1(Y) = \mathbb{Z}/11$ (indeed, $|H_1(Y)| = \det(10_{125}) = 11$), and both $H_2(X)$, $H_2(W)$ are torsion-free, we have

$$0 \rightarrow H_2(X) \oplus H_2(W) \rightarrow H_2(X \cup W) \rightarrow \mathbb{Z}/11 \rightarrow 0,$$

and thus $H_2(X, \mathbb{Z})$ is a subgroup of $L = \mathbb{Z}^{n-7}$ of index 11. Set $m = n - 7 = b_2(X)$, and let $\{u_1, u_2, \dots, u_m\}$ be basis of L in which the form is diagonal, and $u_1 \cdot u_1 = -11$. The vectors $11u_1, 11u_2, \dots, 11u_m$ lie in $H_2(X, \mathbb{Z})$, and generate $H_2(X, \mathbb{Q})$ over \mathbb{Q} .

Now, assume that (X, J) is a Stein filling for ξ , and \mathfrak{s}_J is the corresponding Spin^c structure on X . Let $\bar{\xi}$ be the contact structure on Y conjugate to ξ ; then $\bar{\xi}$ has a Stein filling $(X, -J)$, with $\mathfrak{s}_{-J} = \bar{\mathfrak{s}}_J$ the corresponding Spin^c structure. We have $d_3(\bar{\xi}) = -\frac{1}{2}$, and the classification of contact structures on Y implies that $\bar{\xi}$ is isotopic to ξ . Then by [LM] we must have $c_1(\mathfrak{s}_J) = c_1(\mathfrak{s}_{-J})$, so $c_1(\mathfrak{s}_J) = 0$.

On the other hand, $c_1(\mathfrak{s})$ evaluates as an odd integer on each vector $11u_1, 11u_2, \dots, 11u_m$; it follows that $m = 0$. Then $d_3(\xi) = 0$, which contradicts the calculation $d_3(\xi) = -\frac{1}{2}$.

Example 3.2. Consider the transverse representative of the mirror of the knot 10_{141} given by the braid $\sigma_1^{-4}\sigma_2\sigma_1^3\sigma_2^2$. We consider the family of braids

$$K_r = \sigma_1^{-r}\sigma_2\sigma_1^3\sigma_2^2.$$

The contact surgery description for the corresponding contact structures are shown on Figure 6; the surgery diagrams are quite similar to those in the previous example, but have one extra component. The Kirby calculus moves similar to those in Figure 3 show that the branched double cover is the Seifert fibered space $M(-1; 2/3, 2/3, 1/n)$.

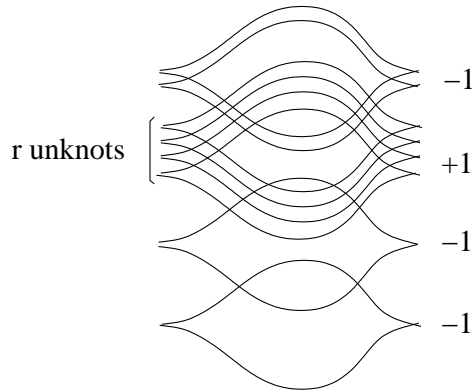


FIGURE 6. The surgery diagrams for the branched double covers of the transverse links K_r .

As before, we can show that all the braids K_r are quasi-alternating. Indeed, we resolve one of the negative crossings to obtain K_{r-1} and a trefoil as two resolutions; we also observe that K_0 is the connected sum of two trefoils. Since $|\det(\text{trefoil})| = 3$, $|\det(K_0)| = 9$

and $|\det(K_r)| = |H_1(M(-1; 2/3, 1/2, 1/n))| = 9 + 3r$, each K_r is quasi-alternating by induction.

Next, we compute $sl(K_r) = 3 - r$, and $s = \sigma(K_r) = 4 - r$; the hypotheses of Corollary 1.3 are therefore satisfied, and all branched covers $\Sigma(K_r)$ are tight contact manifolds.

For the contact structure on the branched cover of K_4 , we compute $d_3 = 0$, which provides no obstruction to Stein fillability. However, for the braid K_6 we get $d_3 = -\frac{1}{2}$. We then argue as in the previous example to show that the branched cover of K_6 is not Stein fillable (and thus the branched double covers of all braids K_r with $r \geq 6$ are not Stein fillable either). Denote $Y = \Sigma(K_6) = M(-1; 2/3, 1/2, 1/6)$; then $-Y$ is the boundary of

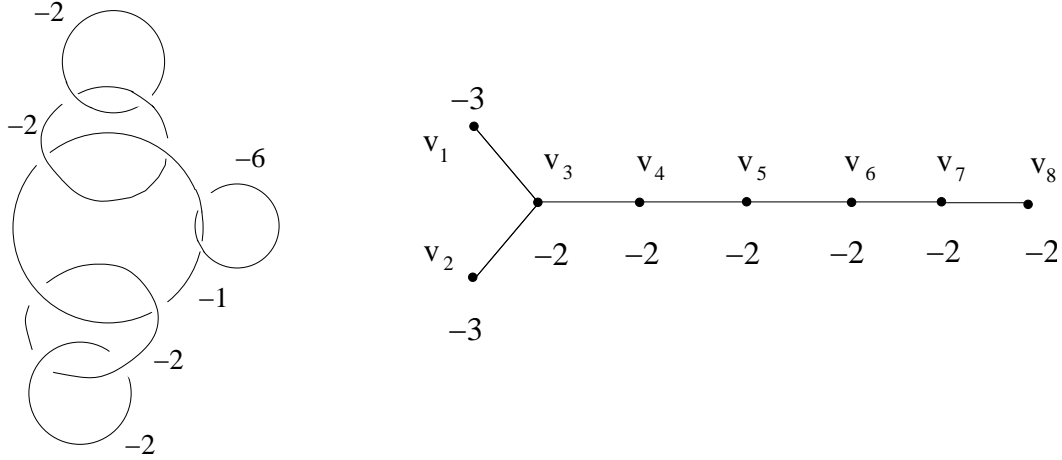


FIGURE 7. A Kirby diagram for $Y = \Sigma(K_6)$ and the plumbing graph for $-Y$ of Example 3.2.

the plumbing W given by the graph on Figure 7. As before, for any symplectic filling X of Y the union $X \cup W$ is a negative-definite closed 4-manifold with the standard diagonal intersection form. Up to changing the signs and the order of the vectors e_i in the diagonal basis, there is a unique embedding of the lattice given by Figure 7 into $\langle -1 \rangle^n$, given by

$$\begin{aligned} v_3 &\mapsto e_1 + e_2, & v_1 &\mapsto -e_1 - e_3 + e_5, & v_2 &\mapsto -e_1 + e_3 + e_4, & v_4 &\mapsto -e_2 + e_6, \\ v_5 &\mapsto -e_6 + e_7, & v_6 &\mapsto -e_7 + e_8, & v_7 &\mapsto -e_8 + e_9, & v_8 &\mapsto -e_9 + e_{10}, \end{aligned}$$

and thus the orthogonal complement of this lattice in $\langle -1 \rangle^n$ is $\langle -9 \rangle \langle -3 \rangle \oplus \langle -1 \rangle^{n-10}$. As in the previous example, the classification of tight contact structures on $M(-1; 2/3, 1/2, 1/6)$ [GLS] implies that our contact structure is isotopic to its conjugate, and so $c_1(X) = 0$ for any Stein filling. Since $|H_1(Y)| = 27$, similar parity argument shows that $b_2(X) = 0$, and so d_3 must be zero, a contradiction.

Remark 3.3. One can try to argue as in [GLS] to investigate symplectic fillability in Examples 3.1 and 3.2: a slightly more involved argument modulo 8 puts further restrictions on the value d_3 for symplectic fillings (with diagonal odd intersection form). However, this gives no obstruction to symplectic fillability of any contact structures in the above two examples.

In the opposite direction, certain tight open books with the punctured torus page and pseudo-Anosov monodromy can be shown to be symplectically fillable as perturbations

of taut foliations [HKM2]. We note that our examples are not pseudo-Anosov, so these results do not apply.

Example 3.4. A transverse representative of the mirror of 10_{130} with the maximal self-linking number is given by the braid $\sigma_1^{-3}\sigma_2\sigma_1^2\sigma_2^2\sigma_3\sigma_2^{-1}\sigma_3$. We consider a family of transverse braids

$$K_r = \sigma_1^{-r}\sigma_2\sigma_1^2\sigma_2^2\sigma_3\sigma_2^{-1}\sigma_3.$$

First, we check that all the underlying links are quasi-alternating. Resolve of the negative crossings among those given by σ_1^{-r} to obtain K_{r-1} as one of the resolutions and the unknot as the other. Observe that K_0 is a two-component alternating link of $\det = 14$ (with 5_2 knot and the unknot as components, linked once). Finally, compute $|\det(K_r)| = 14 + r$ (one way to see this is to compute the size of H_1 of the branched double cover of K_r which is a Seifert fibered space shown on Figure 8).

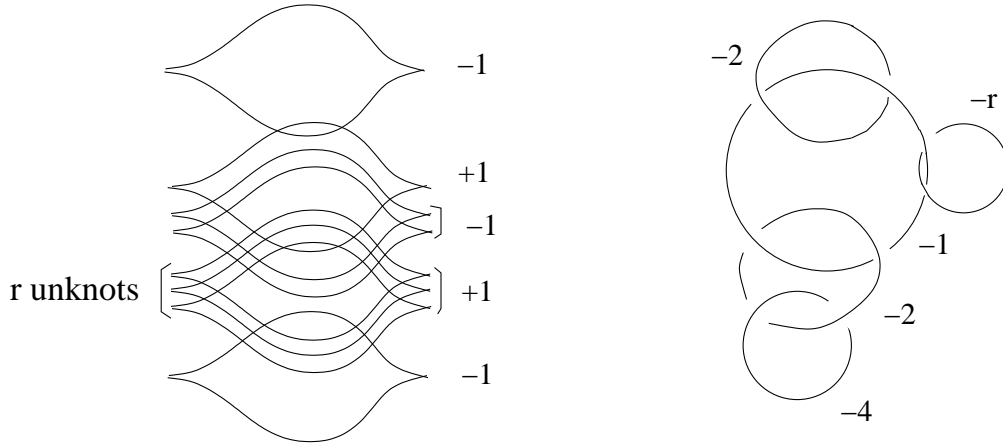


FIGURE 8. The surgery diagrams for the branched double covers of the transverse links K_r .

The hypothesis of Corollary 1.3 holds: $sl(K_r) = 2 - r = \sigma - 1$. Therefore, the branched double covers of the transverse links K_r are all tight.

We do not investigate the fillability question in this case. (One can still try to use the classification of tight contact structures on these small Seifert fibered spaces, the fact they are all L -spaces, and non-vanishing of d_3 for some values of r , but the intersection form for the corresponding plumblings is harder to analyze.)

Remark 3.5. In Examples 3.1 and 3.2, transverse links are 3-braids, and the contact structures on the branched double covers can be given by open books whose page is a once-punctured torus. Tightness of these contact structures can be established by using results of [HKM1] or (easier yet) by rewriting the braids in the “standard” form and using Baldwin’s work [Bal1]. Example 3.4 deals with 4-braids; the page of the corresponding open books is a twice-punctured torus, and known results do not apply.

Remark 3.6. In all of the above examples, we checked explicitly that our families of links are quasi-alternating. In fact a weaker condition, $\text{rk } Kh_{\mathbb{Z}/2}(K) = |\det(K)|$ is sufficient

to ensure that the spectral sequence from Kh to \widehat{HF} collapses at the E^2 stage. For any individual reasonably small knot this can be checked by a computer, for example using Baldwin's `Kh` program [Bal2] that computes the rank of reduced Khovanov homology with $\mathbb{Z}/2$ coefficients. Checking the second condition, $sl(K) = s - 1$, is also routine for $Kh_{\mathbb{Z}/2}$ -thin knots (alternatively, one can use the `Trans` program [Bal2] to check $\psi \neq 0$). Thus tightness of the contact structure on the branched double cover can be established by a computer calculation.

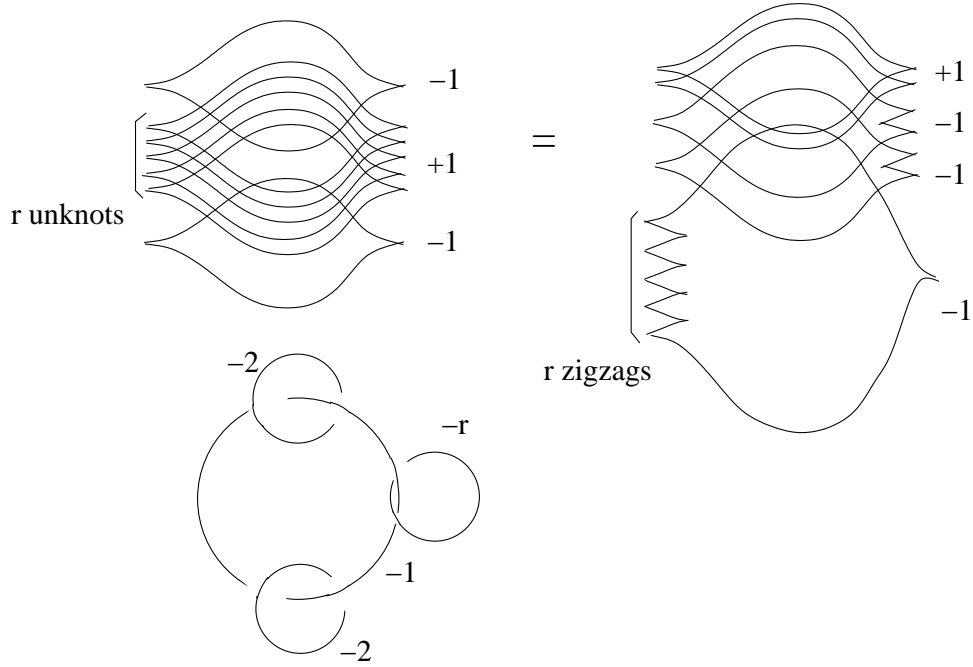


FIGURE 9. The surgery diagrams for the branched double covers of the transverse links of Example 3.7.

Example 3.7. John Baldwin has pointed out that for a family of (non-quasi-alternating) transverse 3-braids

$$K_r = \sigma_1^{-r} \sigma_2 \sigma_1^2 \sigma_2$$

for $r > 2$ the transverse invariant ψ vanishes, but the contact structures on the branched double covers are tight, and have $c(\xi) \neq 0$. The vanishing of ψ can be established by the computer program [Bal2], while non-vanishing of the contact invariant follows from calculations in [HKM1] and [Bal1]. Obviously, Khovanov homology fails to detect tightness in this case (so this is really a non-example), but it is interesting to take a look at the corresponding contact structures. They are given by surgery diagrams on Figure 9, and are very similar to the contact structures from Example 3.1. As the latter are obtained from the former by Legendrian surgery on a knot, the contact structures $\xi(K_r)$ cannot be Stein fillable. As the underlying smooth manifold is $M(-1; 1/2, 1/2, 1/r)$ carries a unique Stein non-fillable contact structure for each r , these contact structures are precisely those considered in [GLS], where most of them are shown to be symplectically non-fillable.

One may wonder whether there is any relation between vanishing of ψ and symplectic non-fillability (although such relation seems quite improbable).

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DEPARTMENT OF MATHEMATICS, SUNY STONY BROOK, STONY BROOK, NY 11794

E-mail address: `olga@math.sunysb.edu`